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Q1.

In cylindrical coordinates : $x = s\cos\phi$, $y = s\sin\phi$, $z = z$

$\therefore$ V is bounded by $z = s^2$ and $z = 8 - s^2$.

So, the range of (s, ϕ, z) are as follows:

$\phi = 0 \to 2\pi$

$s = 0 \to 2$ $\left[\because z = s^2 = 8 - s^2 \Rightarrow s^2 = 4 \text{ and } s \text{ is always +ve}\right]$ [2 +]

$z = s^2 \to 8 - s^2$.

Also, $dx\,dy\,dz = dV = s\,ds\,d\phi\,dz$ [0·5 +]

$$\therefore \iiint_V \sqrt{x^2+y^2}\,dV$$

$$= \iiint_V s \cdot s\,ds\,d\phi\,dz$$

$$= \int_{\phi=0}^{2\pi} d\phi \int_{s=0}^{2} ds\, s^2 \int_{z=s^2}^{8-s^2} dz$$

$$= 2\pi \int_{s=0}^{2} ds\, s^2 (8 - 2s^2)$$

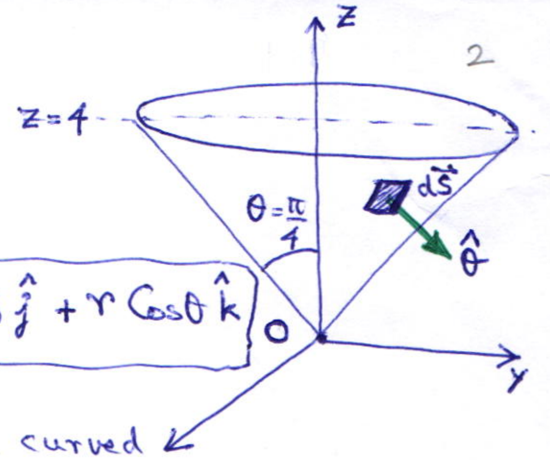
$$= 2\pi \int_{s=0}^{2} ds\, (8s^2 - 2s^4)$$

$$= 2\pi \left[\frac{8s^3}{3} - \frac{2s^5}{5}\right]_{s=0}^{2} = 2\pi\left(\frac{64}{3} - \frac{64}{5}\right) = 128\pi\left(\frac{5-3}{15}\right)$$

$$= \frac{256\pi}{15}.$$ [1·5 +]

CLASSTIME

Q2. $\boxed{\text{CASE I}}$ (r, θ, ϕ) - parametrization



A spherical polar parametrization of an arbitrary point in space is

$$\boxed{\vec{r}(r,\theta,\phi) = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}}$$

Consider a differential area $d\vec{S}$ on the curved surface of the cone who position can be denoted by

$$\boxed{\vec{r}\left(r,\theta=\tfrac{\pi}{4},\phi\right) \equiv \vec{r}\Big|_S = \frac{r}{\sqrt{2}}\left[\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + \hat{k}\right]} \quad [0.5+]$$

The unit vector normal to dS is $\hat{\theta} \equiv \hat{e}_\theta$ at $\theta = \pi/4$

where, $\hat{e}_\theta = \hat{\theta} = \dfrac{\partial\vec{r}}{\partial\theta} \Big/ \left|\dfrac{\partial\vec{r}}{\partial\theta}\right| = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

or, $$\boxed{\hat{e}_\theta\Big|_S \equiv \hat{\theta}\Big|_S = \frac{1}{\sqrt{2}}\left[\cos\phi\,\hat{i} + \sin\phi\,\hat{j} - \hat{k}\right]}$$

[0.5+]

Now, $\vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$

$$\Rightarrow \boxed{\vec{F}\left(r,\theta=\tfrac{\pi}{4},\phi\right) = \vec{F}\Big|_S = 2r^2\cos\phi\,\hat{i} + \frac{r^4}{4}\cos\phi\sin\phi\,\hat{j} + \frac{3}{\sqrt{2}}r\,\hat{k}}$$

[1+]

Then, the total outward flux of $\vec{F}$ through S is

$$\Phi = \iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\hat{\theta})_S\, dS = \iint_{\mathcal{D}} \left(\sqrt{2}\,r^2\cos^2\phi + \frac{r^4}{4\sqrt{2}}\sin^2\phi\cos\phi - \frac{3}{2}r\right) \times \left(\frac{r}{\sqrt{2}}\,dr\,d\phi\right)$$

where, we use, $dS = r\sin\frac{\pi}{4}\,dr\,d\phi$,

with parameter domain: $\boxed{\mathcal{D} = \{r \times \phi \mid r \in [0, 4\sqrt{2}],\ \phi \in [0, 2\pi]\}}$

[1+]

$$\Rightarrow \Phi = \int_{r=0}^{4\sqrt{2}} dr \int_0^{2\pi} d\phi \left[r^3\cos^2\phi + \frac{r^5}{8}\sin^2\phi\cos\phi - \frac{3}{2\sqrt{2}}r^2\right]$$

$$= \int_0^{4\sqrt{2}} dr \left[\pi r^3 + \cancel{\frac{r^5}{24}\sin^3\phi\Big|_0^{2\pi}} - \frac{3\pi}{\sqrt{2}}r^2\right] = \left(\frac{\pi r^4}{4} - \frac{\pi r^3}{\sqrt{2}}\right)_0^{4\sqrt{2}} = 128\pi$$

0

[2+]

Q 3.(a) We know that

$$\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3}\right) = 4\pi\,\delta^3(\vec{r}-\vec{a}) \quad [0.5+]$$

Now, since $\vec{a}$ is inside V; we have

$$\int_V dv\,(r^3+3)\;\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3}\right)$$

$$= \int_V dv\,(r^3+3)\cdot 4\pi\,\delta^3(\vec{r}-\vec{a})$$

$$= 4\pi\,(a^3+3). \quad [0.5+]$$